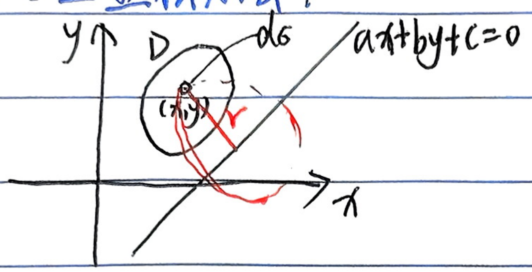
总结 旋转体体积

方法一：微元法

方法二：二重积分法。



取一小区域dσ，该区域绕ax+by+c=0 旋转 ⇒ 甜甜圈状。（面积记为dσ.）

⇒ 拉直 ⇒ 圆柱体.

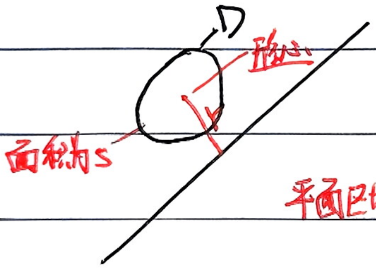
$V_{甜甜圈} = V_{圆柱体} = \underbrace{底面积}_{d\sigma} \times \underbrace{高}_{该点绕线旋转周长}$

$= 2\pi r(x,y)\,d\sigma.$

$r(x,y) = \dfrac{|ax+by+c|}{\sqrt{a^2+b^2}}$ ⇐ r(x,y)为点到直线距离.

$\Rightarrow V = \iint_D 2\pi r(x,y)\,d\sigma.$

方法三、古尔丁定理.

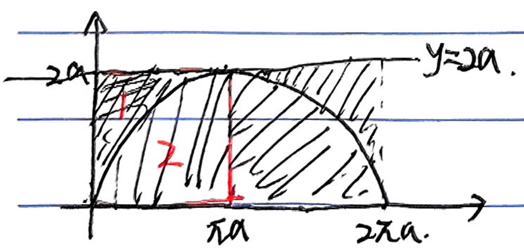


D绕一条不穿过D的直线旋转一周所形成的体积.

$V = 2\pi r\cdot S$

平面区域形心公式 $\left(\dfrac{\iint_D x\,dxdy}{\iint_D dxdy}, \dfrac{\iint_D y\,dxdy}{\iint_D dxdy}\right)$.

例：660题 $\boxed{165}$ 摆线 $x=a(t-\sin t)$, $y=a(1-\cos t)$ $(0\le t\le 2\pi)$ 与x轴围成图形绕 $y=2a$ 旋转一周得旋转体体积 $V=$



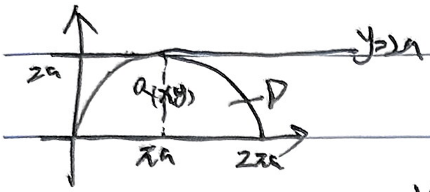
$\boxed{法一}$ 微元法. $V = 2(V_2 - V_1) = 2(V_{圆柱} - V_{曲边A}).$

$= 2\left[\pi(2a)^2\cdot\pi a - \int_0^{\pi a}\pi(2a-y)^2\cdot dx\right]$

$= 2\times\left(4\pi^2a^3 - \dfrac{\pi^2a^3}{2}\right)$

$= 7\pi^2a^3$

法二 二重积分法.



$d\sigma = dxdy$.

$dV = 2\pi r\, d\sigma$. $\quad r = (2a-y)$.

$V = \iint_D 2\pi(2a-y)\,dxdy$.

$= 2\pi\int_0^{2\pi a} dx\int_0^{y}(2a-y)dy$. $\quad$ 其中 $\int_0^{y}(2a-y)dy = 2ay-\frac{1}{2}y^2\Big|_0^{y}$

$= 2ay-\frac{1}{2}y^2$.

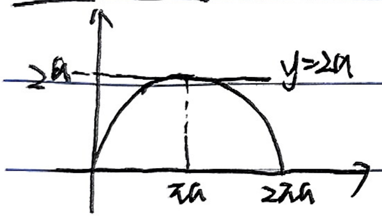
$= 2\pi\int_0^{2\pi a}(2ay-\frac{1}{2}y^2)\,dx$.

$= \pi\int_0^{2\pi a} y(4a-y)\,dx$. 代入 $x=a(t-\sin t)$ $y=a(1-\cos t)$

$= \pi\int_0^{2\pi} a(1-\cos t)(3a+a\cos t)\,d\,a(t-\sin t)$

$= 7\pi^2 a^3$.

法三 古尔丁定理.



$V = 2\pi d S$.

$\begin{cases} x=a(t-\sin t) \\ y=a(1-\cos t) \end{cases}$

形心 r 为 $(\pi a, \bar{y})$.

$\bar{y} = \dfrac{\iint_D y\,dxdy}{\iint_D dxdy}$

$S = \iint_D dxdy = \int_0^{2\pi a}dx\int_0^{y}dy$.

$= \int_0^{2\pi a} y\,dx$

$= 3\pi a^2$.

$\iint_D y\,dxdy = \frac{5}{2}a^3\pi$

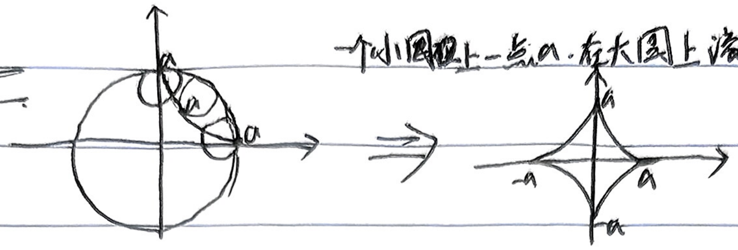
$\Rightarrow \bar{y} = \dfrac{\frac{5}{2}a^3\pi}{3a^2\pi} = \frac{5}{6}a$. $\quad \Rightarrow r(\pi a, \frac{5}{6}a)$.

$\therefore V = 2\pi\cdot\underbrace{(2a-\tfrac{5}{6}a)}_{\frac{7}{6}a}\cdot 3\pi a^2 = 7\pi^2 a^3$.

总结. 弧长&侧面积&星形线　　注. 参数方程. 极坐标方程 → 换元法 → 一一对应.

星形线

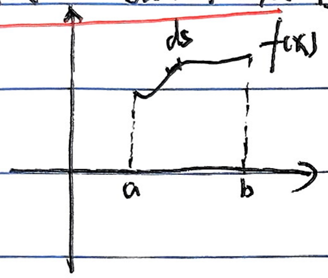
一个小圆上一点a. 在大圆上滚动走过的路径.



星形线方程 $\begin{cases} x = a\cos^3 t \\ y = a\sin^3 t \end{cases} \Longleftrightarrow x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$

总结：直角坐标系下的 弧长公式 旋转体侧面积公式.

弧长公式：$S = \int_a^b \sqrt{1+y'^2}\,dx$. $\boxed{b>a}$



ds 可看成直线.

ds, dx, dy　$S = \int_a^b ds$

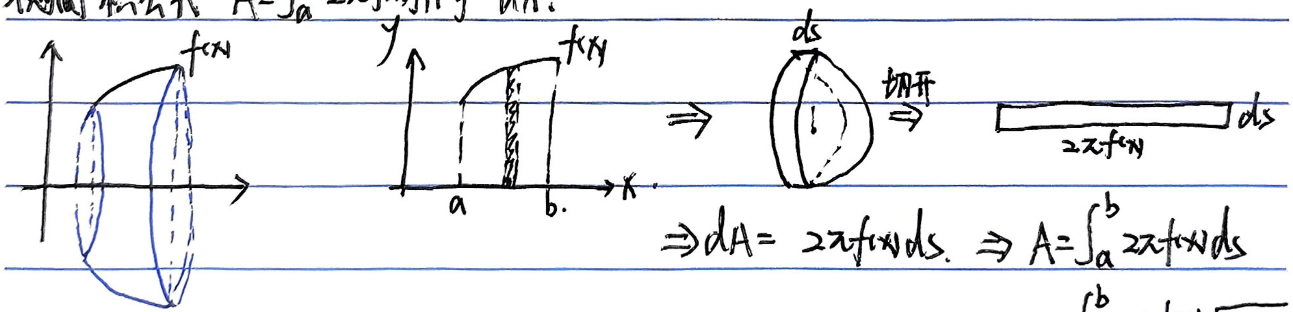
$\Downarrow$

$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{(dx)^2\left[1 + \left(\frac{dy}{dx}\right)^2\right]}$

$= \sqrt{1+y'^2}\,|dx| = \sqrt{1+y'^2}\,dx$

$\Rightarrow S = \int_a^b \sqrt{1+y'^2}\,|dx|$ $\begin{cases} a>b & \int_a^b \sqrt{1+y'^2}\,(-dx) = \int_b^a \sqrt{1+y'^2}\,dx. \\ b>a & \int_a^b \sqrt{1+y'^2}\,dx. \end{cases}$

侧面积公式 $A = \int_a^b 2\pi f(x)\sqrt{1+y'^2}\,dx$.

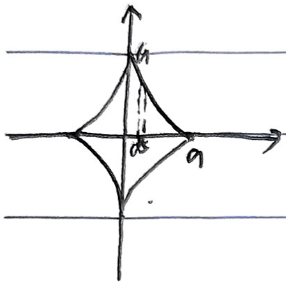


$\Rightarrow dA = 2\pi f(x)\,ds.$ $\Rightarrow A = \int_a^b 2\pi f(x)\,ds$

$= \int_a^b 2\pi f(x)\sqrt{1+f'^2(x)}\,dx.$ $\boxed{b>a}$

例. 6(1)题. $\boxed{(6)}$

星形线 $\begin{cases} x = a\cos^3 t \\ y = a\sin^3 t \end{cases}$. 求围成面积A、弧长L、绕x旋转体积V、旋转体侧面积S.



围成面积 $A = 4\int_0^a y\,dx = 4\int_{\frac{\pi}{2}}^0 a\sin^3 t\,d\,a\cos^3 t$

$= 4\int_0^{\frac{\pi}{2}} 3a^2\cos^2 \sin^4 t\,dt$

$= 12a^2\int_0^{\frac{\pi}{2}} (1-\sin^2 t)\sin^4 t\,dt$

$= 12a^2\left[\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2} - \frac{5}{6}\times\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2}\right]$

$= \frac{3}{8}\pi a^2$

弧长L. $S = \int_a^b \sqrt{1+y'^2}\,dx$.

$\cancel{S = 4\int_{\frac{\pi}{2}}^{a} \sqrt{1+(3a\sin^2 t\cos t)^2}\,d\,a\cos^3 t}$ $y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3a\sin^2 t\cos t}{-3a\cos^2 t\sin t} = -\tan t$.

$S = 4\int_{\frac{\pi}{2}}^0 \sec t\cdot 3a\cos^2 t\sin t\,dt = 12a\int_0^{\frac{\pi}{2}} \sin t\cos t\,dt = 6a$.

绕x体积V $V = 2\int_0^a \pi y^2\,dx = 2\pi\int_0^{\frac{\pi}{2}} a^2\sin^6 t\cdot 3a\cos^2 t\sin t\,dt$

$= 6a^3\pi\int_0^{\frac{\pi}{2}} \sin^7 t\cdot(1-\sin^2 t)\,dt$

$= 6a^3\pi\left[\frac{6}{7}\times\frac{4}{5}\times\frac{2}{3} - \frac{8}{9}\times\frac{6}{7}\times\frac{4}{5}\times\frac{2}{3}\right] = \frac{32}{105}\pi a^3$

侧面积S $S = 2\int_0^a 2\pi y\,ds = 2\int_0^a 2\pi y\sqrt{1+y'^2}\,dx$.

$= 2\int_0^{\frac{\pi}{2}} 2\pi a\sin^3 t\cdot\sec t\cdot 3a\cos^2 t\sin t\,dt$.

$= 12\pi a^2\int_0^{\frac{\pi}{2}} \sin^4 t\cos t\,dt = 12\pi a^2\cdot\frac{1}{5}\sin^5 t\Big|_0^{\frac{\pi}{2}} = \frac{12}{5}\pi a^2$.